Announcements

1) Course progression

2.4, 2.6 – 2.8, 3.4,

3.2, 3.3, 4.1, 4.2

We proved: Let $v_1, \dots, v_m \in \mathbb{C}^n$

and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \in M_{n \times m}(\mathbb{C})$.

Then

1) $\{v_1, \dots, v_m\}$ is linearly independent iff the echelon form of A has a pivot in every column.

2) $\{v_1, \dots, v_m\}$ is spanning iff the echelon form of A has a pivot in every row.

3) $\{v_1, \dots, v_m\}$ is a basis iff the echelon form of A has a pivot in every row & column.

Corollary: Let S be a subset of $\mathbb{C}^n$ (or $\mathbb{R}^n$).

Then

1) If S is linearly independent, then $|S| \leq n$.

2) If S is spanning (generating), then $|S| \geq n$.

Proof: 1) Suppose S is linearly independent and $|S| > n$. Suppose, in addition, that S is finite. Consider the $n \times |S|$ matrix A formed by using the vectors in S as columns.

By our previous results, A has a pivot in every column, so A has $|S|$ pivots. But the number of pivots of A is bounded above by $\min(n, |S|) = n$, contradiction.

Therefore $|S| \leq n$.

Now suppose $|S|$ is infinite. Then $\text{span}(S)$ is an infinite dimensional subspace of $\mathbb{C}^n$ (or $\mathbb{R}^n$), which is a contradiction since if $W \subseteq \mathbb{C}^n$ is a subspace, $\dim(W)$ must be finite.

2) Similar to 1), using instead the fact that A has a pivot in every row.

$\square$

Example 1:

a) Let $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \right\} \subseteq \mathbb{C}^3$.

Then S does not span $\mathbb{C}^3$

$(|S| = 2 < 3)$.

However, S is linearly independent.

b) Let

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\} \subseteq \mathbb{C}^2 .$$

Then S is not linearly independent ($|S| = 4 > 2$).

However, S is spanning.

Theorem: (invertibility)

Let $A \in M_n(\mathbb{C})$. Then A is invertible iff the columns of A form a basis for $\mathbb{C}^n$.

proof: $\Rightarrow)$ Suppose A is invertible. Consider the standard basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$.

Then $Ae_1 = 1^{st}$ column of A

$Ae_2 = 2^{nd}$ column of A

$\vdots$

$Ae_n = n^{th}$ column of A.

All we need do is show

$$\{Ae_1, Ae_2, \ldots, Ae_n\}$$

is linearly independent.

Suppose

$$\sum_{i=1}^{n} \alpha_i (Ae_i) = 0$$

for some scalars $\alpha_1, \dots, \alpha_n$.

By linearity,

$$0 = \sum_{i=1}^{n} \alpha_i (Ae_i) = \sum_{i=1}^{n} A(\alpha_i e_i)$$

$$= A\left(\sum_{i=1}^{n} \alpha_i e_i\right)$$

Then with $x = \sum_{i=1}^{n} \alpha_i e_i$,

$$Ax = 0 \Rightarrow x \in \ker(A).$$

But A invertible

$\Rightarrow \ker(A) = 0$,

so $x = 0$.

Therefore

$$0 = \sum_{i=1}^{n} \alpha_i e_i$$

$\Rightarrow \alpha_i = 0 \quad \forall\ 1 \leq i \leq n$

Since the set $\{e_i\}_{i=1}^{n}$ is a basis.

We then have that

$\{Ae_i\}_{i=1}^{n}$ is linearly independent $\Rightarrow$

$\{Ae_i\}_{i=1}^{n}$ is a basis for $\mathbb{C}^n$.

$\Leftarrow$ Suppose the columns of A form a basis for $\mathbb{C}^n$. Then by our previous results, the echelon form of A has a pivot in every row & column.

Let A_0 = echelon form of A.

Then A_0 is upper-triangular with diagonal $\{x_1, x_2, \ldots, x_n\}$ nonzero.

Claim: by using further row operations, we can row-reduce A_0 (and hence, A) to the identity matrix.

You can prove this via induction. So then

$\exists$ E_1, E_2 products of elementary matrices with

$$A_0 = E_1 A,$$

$$I_n = E_2 A_0$$
$$= (E_2 E_1) A$$

$\Rightarrow E_2 E_1$ is the inverse of A.

$\square$

Corollary: (finding A^{-1})

Let $A \in M_n(\mathbb{C})$, suppose A is invertible. Then to find A^{-1}, row reduce the $n \times 2n$ matrix

$$(A \mid I_n)$$

until you see

$$(I_n \mid B).$$

Then $B = A^{-1}$.

proof: Use the proof of the previous result to multiply $(A \mid I_n)$ by elementary matrices until you see $(I_n \mid B)$. By definition, $B = A^{-1}$.

$\square$

Theorem: $(Ax=b)$

Let $A \in M_{m\times n}(\mathbb{C})$, $x \in \mathbb{C}^n$, $b \in \mathbb{C}^m$. Then all solutions of $Ax=b$ may be expressed as $y+x_0$ where y is some solution to $Ax=b$ and x_0 is a solution to $Ax_0=0$.