

Announcements

1) Course progression

2.4, 2.6-2.8, 3.4,

3.2, 3.3, 4.1, 4.2

We proved: Let $v_1, \dots, v_m \in \mathbb{C}^n$

and $A = [v_1 \ v_2 \ \dots \ v_m] \in M_{n \times m}(\mathbb{C})$.

Then

1) $\{v_1, \dots, v_m\}$ is linearly independent iff the echelon form of A has a pivot in every column.

2) $\{v_1, \dots, v_m\}$ is **spanning**
iff the echelon form of
A has a pivot in every **row**.

3) $\{v_1, \dots, v_m\}$ is a **basis**
iff the echelon form of
A has a pivot in every
row + column.

Corollary: Let S be a subset of \mathbb{C}^n (or \mathbb{R}^n).

Then

- 1) If S is linearly independent, then $|S| \leq n$.
- 2) If S is spanning (generating), then $|S| \geq n$.

Proof: 1) Suppose S is linearly independent and $|S| > n$. Suppose, in addition, that S is finite. Consider the $n \times |S|$ matrix A formed by using the vectors in S as columns.

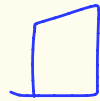
By our previous results,
A has a pivot in every
column, so A has $|S|$
pivots. But the number
of pivots of A is
bounded above by

$\min(n, |S|) = n$, contradiction.

Therefore $|S| \leq n$.

Now suppose $|S|$ is infinite. Then $\text{span}(S)$ is an infinite dimensional subspace of \mathbb{C}^n (or \mathbb{R}^n), which is a contradiction since if $W \subseteq \mathbb{C}^n$ is a subspace, $\dim(W)$ must be finite.

2) Similar to 1), using
instead the fact that
 A has a pivot in
every row.



Example 1:

$$\text{a) Let } S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \right\} \\ \subseteq \mathbb{C}^3.$$

Then S does not span \mathbb{C}^3
($|S| = 2 < 3$).

However, S is linearly
independent.

b) Let

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\} \\ \subseteq \mathbb{C}^2.$$

Then S is not linearly independent ($|S| = 4 > 2$).

However, S is spanning.

Theorem: (invertibility)

Let $A \in M_n(\mathbb{C})$. Then

A is invertible iff

the columns of A form

a basis for \mathbb{C}^n .

proof: \Rightarrow Suppose A is

invertible. Consider the

standard basis $\{e_1, \dots, e_n\}$

for \mathbb{C}^n .

Then $Ae_1 = 1^{\text{st}}$ column of A

$Ae_2 = 2^{\text{nd}}$ column of A

⋮

$Ae_n = n^{\text{th}}$ column of A .

All we need do is show

$\{Ae_1, Ae_2, \dots, Ae_n\}$

is linearly independent.

Suppose

$$\sum_{i=1}^n \alpha_i (Ae_i) = 0$$

for some scalars $\alpha_1, \dots, \alpha_n$.

By linearity,

$$\begin{aligned} 0 &= \sum_{i=1}^n \alpha_i (Ae_i) = \sum_{i=1}^n A(\alpha_i e_i) \\ &= A \left(\sum_{i=1}^n \alpha_i e_i \right) \end{aligned}$$

Then with $x = \sum_{i=1}^n \alpha_i e_i$,

$$Ax = 0 \Rightarrow x \in \ker(A).$$

But A invertible

$$\Rightarrow \ker(A) = 0,$$

$$\text{so } x = 0.$$

Therefore

$$0 = \sum_{i=1}^n \alpha_i e_i$$

$$\Rightarrow \alpha_i = 0 \quad \forall 1 \leq i \leq n$$

Since the set $\{e_i\}_{i=1}^n$ is a basis.

We then have that

$\{Ae_i\}_{i=1}^n$ is linearly
independent \Rightarrow

$\{Ae_i\}_{i=1}^n$ is a basis
for \mathbb{C}^n .

⊆ Suppose the columns of A form a basis for \mathbb{C}^n . Then by our previous results, the echelon form of A has a pivot in every row & column.

Let $A_0 =$ echelon form of A .

Then A_0 is upper-triangular
with diagonal

$$\{x_1, x_2, \dots, x_n\}$$

nonzero.

Claim: by using further
row operations, we can
row-reduce A_0 (and hence,
 A) to the identity
matrix.

You can prove this via induction. So then

$\exists E_1, E_2$ products of elementary matrices with

$$A_0 = E_1 A,$$

$$I_n = E_2 A_0$$

$$= (E_2 E_1) A$$

$\Rightarrow E_2 E_1$ is the inverse of A .

□

Corollary: (finding A^{-1})

Let $A \in M_n(\mathbb{C})$, suppose A is invertible. Then to find A^{-1} , row reduce the $n \times 2n$ matrix

$$(A \mid I_n) \text{ until}$$

you see

$$(I_n \mid B). \text{ Then } B = A^{-1}.$$

proof:

Use the proof of
the previous result
to multiply $(A | I_n)$
by elementary matrices
until you see $(I_n | B)$.

By definition, $B = A^{-1}$.

□

Theorem: ($Ax=b$)

Let $A \in M_{m \times n}(\mathbb{C})$, $x \in \mathbb{C}^n$,
 $b \in \mathbb{C}^m$. Then all solutions
of $Ax=b$ may be
expressed as $y+x_0$ where
 y is **some** solution to $Ax=b$
and x_0 is a solution to
 $Ax_0=0$.